Generalization of the double reduction theory

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Abstract

In a recent work [1, 2] Sjöberg remarked that generalization of the double reduction theory to partial differential equations of higher dimensions is still an open problem. In this note we have attempted to provide this generalization to find invariant solution for a non linear system of qth order partial differential equations with n independent and m dependent variables provided that the non linear system of partial differential equations admits a nontrivial conserved form which has at least one associated symmetry in every reduction. In order to give an application of the procedure we apply it to the nonlinear (2 + 1) wave equation for arbitrary function $f(u)$ and $g(u)$.

Key words: Double reduction theory, Conservation laws, Associated symmetry, Invariant solutions

1 Introduction

Applying a Lie point or Lie-Bäcklund symmetry generator to a conserved vector provide either (1) Conservation law associated with that symmetry or (2) Conservation law that may be trivial, known already or new. A pioneering work in this direction was published by Kara et. al [5, 6]. Sjöberg later showed that [1, 2] when the generated conserved vector is null, i.e. the symmetry is associated with the conserved vector (association defined as in [5]), a double reduction is possible for PDEs with two independent variables. In this double reduction the PDE of order q is reduced to an ODE of order $(q-1)$. Thus the use of one symmetry associated with a conservation law leads to two reductions, *the first being a reduction of the number of independent variables* and *the second being a reduction of the order of the DE.* Sjöberg also constructed the reduction formula for PDEs with two independent variables which transform the conserved form of the PDE to a reduced conserved form via an associated symmetry. Application of this method to the linear heat, the BBM and the sine-Gordon equation and a system of differential equations from one dimensional gas dynamics are given [1]. The double reduction theory says that a PDE of order q with two independent and m dependent variables, which admits a nontrivial conserved form that has at least one associated symmetry can be reduced to an ODE of order $(q-1)$.
In her papers [1, 2] Sjöberg opines that generalizing the double reduction theory to PDEs of higher dimensions is still an open problem and it is not clear how to overcome the problem when not all derivatives of non-local variables are known explicitly. Further calculations for higher dimensions are quite tedious and cumbersome. There do not exist enough examples of potential symmetries and symmetries with associated

conservation laws for higher dimensional PDEs so that the complexity of this problem can be demonstrated. And much work is needed to generalize (if possible) the theory to PDEs with more than two independent variables.

In this article we discuss *a generalization of the double reduction theory* with n independent variables by showing that a non linear system of qth order PDEs with n independent and m dependent variables, which admits a nontrivial conserved form that has at least one associated symmetry in every reduction from the n reductions(the first step of double reduction) can be reduced to a non linear system of $(q-1)$th order ODEs. In order to solve this we use two main steps: (a) Generalize the reduction formula of Sjöberg in [1] from two independent variable to n independent variables and (b) prove that the conserved form of PDEs with n independent variables can be transformed to a reduced conserved form via an associated symmetry. Finally we apply the generalized double reduction to the nonlinear (2 + 1) wave equation for arbitrary function $f(u)$ and $g(u)$ to obtain invariant solution.

2 The Fundamental Theorem of double reduction

Consider the qth-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$

$$E^{\alpha}(x, u, u_{(1)}, ..., u_{(q)}) = 0, \quad \alpha = 1, ..., m\ , \tag{2.1}$$

where $u_{(1)}, u_{(2)}, ..., u_{(q)}$ denote the collections of all first, second,..., q th-order partial derivatives, i.e., $u_i^{\alpha} = D_i(u^{\alpha}), u_{ij}^{\alpha} = D_j D_i(u^{\alpha})$,...respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}} + ..., \quad i = 1, ..., n\ , \tag{2.2}$$

in which the summation convention is used.

The following definitions are well-known (see, e.g. [3, 4, 5]).

The Lie-Bäcklund operator is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^{\alpha}\frac{\partial}{\partial u^{\alpha}} \quad \xi^i, \eta^{\alpha} \in A\ , \tag{2.3}$$

where A is the space of *differential functions*. The operator (2.3) is an abbreviated form of the infinite formal sum

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^{\alpha}\frac{\partial}{\partial u^{\alpha}} + \sum_{s\geq 1}\zeta^{\alpha}_{i_1 i_2 ... i_s}\frac{\partial}{\partial u^{\alpha}_{i_1 i_2 ... i_s}}, \tag{2.4}$$

where the additional coefficients are determined uniquely by the prolongation formulae,

$$\begin{aligned} &\zeta_i^{\alpha} = D_i(W^{\alpha}) + \xi^j u_{ij}^{\alpha} \\ &\zeta^{\alpha}_{i_1 ... i_s} = D_{i_1}...D_{i_s}(W^{\alpha}) + \xi^j u^{\alpha}_{j i_1 ... i_s}, \quad s > 1, \end{aligned} \tag{2.5}$$

in which W^{α} is the *Lie characteristic function,*

$$W^{\alpha} = \eta^{\alpha} - \xi^j u_j^{\alpha}. \tag{2.6}$$

The n-tuple vector $T = (T^1, T^2, ..., T^n)$, $T^j \in A,\ j = 1, ..., n$ is a conserved vector of (2.1) if T^i satisfies

$$D_i T^i \mid_{(2.1)} = 0. \tag{2.7}$$

A Lie-Bäcklund symmetry generator X of the form (2.4) is associated with a conserved vector T of the system (2.1) if X and T satisfy the relations

$$[T^i,X]=X(T^i)+T^iD_k(\xi^k)-T^kD_k(\xi^i)=0, \quad i=1,...,n. \tag{2.8}$$

Theorem 2.1 [3, 4] Suppose that X is any Lie-Bäcklund symmetry of (2.1) and $T^i, i=1,...,n$ are the components of conserved vector of (2.1).Then

$$T^{*i}=[T^i,X]=X(T^i)+T^iD_j\xi^j-T^jD_j\xi^i, \quad i=1,...,n. \tag{2.9}$$

constitute the components of a conserved vector of (2.1) ,i.e.
$D_iT^{*i}\mid_{(2.1)}=0$
Theorem 2.2 [7] Suppose $D_iT^i=0$ is a conservation law of PDE system (2.1). Under the contact transformation, there exist functions $\tilde{T}^i$ such that $J\ D_iT^i=\tilde{D}_i\tilde{T}^i$ where $\tilde{T}^i$ is given explicitly in terms of the determinant obtained through replacing the ith row of the Jacobian determinant by$[T^1,T^2,...,T^n]$, where

$$J=\begin{vmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \dots & \tilde{D}_1x_n \\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \dots & \tilde{D}_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \dots & \tilde{D}_nx_n \end{vmatrix} \tag{2.10}$$

Theorem 2.3 Suppose $D_iT^i=0$ is a conservation law of PDE system (2.1). Under the contact transformation, there exist functions $\tilde{T}^i$ such that $J\ D_iT^i=\tilde{D}_i\tilde{T}^i$ where $\tilde{T}^i$ is given explicitly in terms of

$$\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix}=J(A^{-1})^T\begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J\begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}=A^T\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} \quad , \tag{2.11}$$

where

$$A=\begin{pmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \dots & \tilde{D}_1x_n \\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \dots & \tilde{D}_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \dots & \tilde{D}_nx_n \end{pmatrix}, \quad A^{-1}=\begin{pmatrix} D_1\tilde{x_1} & D_1\tilde{x_2} & \dots & D_1\tilde{x_n} \\ D_2\tilde{x_1} & D_2\tilde{x_2} & \dots & D_2\tilde{x_n} \\ \vdots & \vdots & \vdots & \vdots \\ D_n\tilde{x_1} & D_n\tilde{x_2} & \dots & D_n\tilde{x_n} \end{pmatrix} \tag{2.12}$$

and $J=det(A)$.
Proof :

Using theorem 2.2 we can write

$$\tilde{T}^1=\begin{vmatrix} T_1 & T_2 & \dots & T_n \\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \dots & \tilde{D}_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \dots & \tilde{D}_nx_n \end{vmatrix}=\frac{1}{J}\begin{vmatrix} J\ T_1 & \tilde{D}_2x_1 & \dots & \tilde{D}_nx_1 \\ J\ T_2 & \tilde{D}_2x_2 & \dots & \tilde{D}_nx_2 \\ \vdots & \vdots & \vdots & \vdots \\ J\ T_n & \tilde{D}_2x_n & \dots & \tilde{D}_nx_n \end{vmatrix}, \tag{2.13}$$

$$\tilde{T}^2=\begin{vmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \dots & \tilde{D}_1x_n \\ T_1 & T_2 & \dots & T_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \dots & \tilde{D}_nx_n \end{vmatrix}=\frac{1}{J}\begin{vmatrix} \tilde{D}_1x_1 & J\ T_1 & \dots & \tilde{D}_nx_1 \\ \tilde{D}_1x_2 & J\ T_2 & \dots & \tilde{D}_nx_2 \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_1x_n & J\ T_n & \dots & \tilde{D}_nx_n \end{vmatrix}, \tag{2.14}$$

$$\tilde{T}^n = \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ T_1 & T_2 & \dots & T_n \end{vmatrix} = \frac{1}{J} \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_2 x_1 & \dots & J\,T_1 \\ \tilde{D}_1 x_2 & \tilde{D}_2 x_2 & \dots & J\,T_2 \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_1 x_n & \tilde{D}_2 x_n & \dots & J\,T_n \end{vmatrix}. \tag{2.15}$$

Since

$$J = \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \dots & \tilde{D}_n x_n \end{vmatrix} = \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_2 x_1 & \dots & \tilde{D}_n x_1 \\ \tilde{D}_1 x_2 & \tilde{D}_2 x_2 & \dots & \tilde{D}_n x_2 \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_1 x_n & \tilde{D}_2 x_n & \dots & \tilde{D}_n x_n \end{vmatrix} = \left|A^T\right|, \tag{2.16}$$

one can use the Cramer's rule to find that $\tilde{T}^1, \tilde{T}^2, \dots, \tilde{T}^n$ can be written as follows:

$$\begin{pmatrix} J\,T^1 \\ J\,T^2 \\ \vdots \\ J\,T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix}. \tag{2.17}$$

Lastly, one can easily see that

$$AA^{-1} = I. \tag{2.18}$$

Lemma 2.1

Consider n independent variables $x = (x^1, x^2, ..., x^n)$, m dependent variables $u = (u^1, u^2, ..., u^m)$ and the change of independent variables $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, ..., \tilde{x}^n)$, then any vector $f(x, u, u_1) = (f^1, f^2, ..., f^n)$ must satisfy the following identity

$$\begin{bmatrix} \tilde{D}_1 & \tilde{D}_1 & \dots & \tilde{D}_1 \\ \tilde{D}_2 & \tilde{D}_2 & \dots & \tilde{D}_2 \\ \vdots & \vdots & \dots & \vdots \\ \tilde{D}_n & \tilde{D}_n & \dots & \tilde{D}_n \end{bmatrix} \begin{pmatrix} f^1 & f^2 & \dots & f^n \\ f^1 & f^2 & \dots & f^n \\ \vdots & \vdots & \dots & \vdots \\ f^1 & f^2 & \dots & f^n \end{pmatrix} = A \begin{bmatrix} D_1 & D_1 & \dots & D_1 \\ D_2 & D_2 & \dots & D_2 \\ \vdots & \vdots & \dots & \vdots \\ D_n & D_n & \dots & D_n \end{bmatrix} \begin{pmatrix} f^1 & f^2 & \dots & f^n \\ f^1 & f^2 & \dots & f^n \\ \vdots & \vdots & \dots & \vdots \\ f^1 & f^2 & \dots & f^n \end{pmatrix}, \tag{2.19}$$

where

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \dots & \tilde{D}_n x_n \end{pmatrix} \tag{2.20}$$

Proof :

Since

$$\tilde{D}_i f^j = \tilde{D}_i x_k D_k f^j, \quad i, j = 1, ..n, \tag{2.21}$$

then

$$\begin{pmatrix} \tilde{D}_1 f^1 & \tilde{D}_1 f^2 & \dots & \tilde{D}_1 f^n \\ \tilde{D}_2 f^1 & \tilde{D}_2 f^2 & \dots & \tilde{D}_2 f^n \\ \vdots & \vdots & \dots & \vdots \\ \tilde{D}_n f^1 & \tilde{D}_n f^2 & \dots & \tilde{D}_n f^n \end{pmatrix} = A \begin{pmatrix} D_1 f^1 & D_1 f^2 & \dots & D_1 f^n \\ D_2 f^1 & D_2 f^2 & \dots & D_2 f^n \\ \vdots & \vdots & \dots & \vdots \\ D_n f^1 & D_n f^2 & \dots & D_n f^n \end{pmatrix} \tag{2.22}$$

Theorem 2.4 (Fundamental Theorem of double reduction).
Suppose $D_iT^i = 0$ is a conservation law of PDE system (2.1). Under the similarity transformation of a symmetry X of the form (2.4) for the PDE, there exist functions $\tilde{T}^i$ such that X is still a symmetry for the PDE $\tilde{D}_i\tilde{T}^i = 0$ and

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [T^1, X] \\ [T^2, X] \\ \vdots \\ [T^n, X] \end{pmatrix}, \tag{2.23}$$

where

$$A = \begin{pmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \dots & \tilde{D}_1x_n \\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \dots & \tilde{D}_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \dots & \tilde{D}_nx_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1\tilde{x_1} & D_1\tilde{x_2} & \dots & D_1\tilde{x_n} \\ D_2\tilde{x_1} & D_2\tilde{x_2} & \dots & D_2\tilde{x_n} \\ \vdots & \vdots & \vdots & \vdots \\ D_n\tilde{x_1} & D_n\tilde{x_2} & \dots & D_n\tilde{x_n} \end{pmatrix} \tag{2.24}$$

and $J = det(A)$.

Proof :

By the above theorem there exist functions $\tilde{T}^i$ such that $J\ D_iT^i = \tilde{D}_i\tilde{T}^i$ and

$$\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} \tag{2.25}$$

Then X is a symmetry for the PDE $\tilde{D}_i\tilde{T}^i = 0$, because $X(J)\ D_iT^i + J\ X(D_iT^i) = X(\tilde{D}_i\tilde{T}^i)$ and

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} XT^1 \\ XT^2 \\ \vdots \\ XT^n \end{pmatrix} + JX((A^{-1})^T) \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} + X(J)(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}. \tag{2.26}$$

Since $J = det(A)$, then

$$X(J) = \begin{vmatrix} \tilde{D}_1\xi^1 & \tilde{D}_1\xi^2 & \dots & \tilde{D}_1\xi^n \\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \dots & \tilde{D}_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \dots & \tilde{D}_nx_n \end{vmatrix} + \begin{vmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \dots & \tilde{D}_1x_n \\ \tilde{D}_2\xi^1 & \tilde{D}_2\xi^2 & \dots & \tilde{D}_2\xi^n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \dots & \tilde{D}_nx_n \end{vmatrix} + \dots + \begin{vmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \dots & \tilde{D}_1x_n \\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \dots & \tilde{D}_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n\xi^1 & \tilde{D}_n\xi^2 & \dots & \tilde{D}_n\xi^n \end{vmatrix} \tag{2.27}$$

Let ζ_{ij} denote the cofactor of $\tilde{D}_i\xi^j$, then it is the cofactor of $\tilde{D}_ix_j$ for the matrix A. Thus

$$X(J) = \tilde{D}_i\xi^j \ \ \zeta_{ij} = D_k\xi^j \ \ \tilde{D}_ix_k \ \ \zeta_{ij} = D_k\xi^j \ \ \delta_{jk} \ \ J. \tag{2.28}$$

Since $\tilde{D}_ix_k \ \ \zeta_{ij} = \delta_{jk} \ \ J$ for every fixed j where δ_{jk} is the Kronecker delta, then

$$X(J) = J(D_1\xi^1 + D_2\xi^2 + \dots + D_n\xi^n) \tag{2.29}$$

Now using the previous lemma one gets,

$$\left[\begin{array}{cccc} \tilde{D}_1 & \tilde{D}_1 & \dots & \tilde{D}_1 \\ \tilde{D}_2 & \tilde{D}_2 & \dots & \tilde{D}_2 \\ \vdots & \vdots & \dots & \vdots \\ \tilde{D}_n & \tilde{D}_n & \dots & \tilde{D}_n \end{array}\right] \left(\begin{array}{cccc} \xi^1 & \xi^2 & \dots & \xi^n \\ \xi^1 & \xi^2 & \dots & \xi^n \\ \vdots & \vdots & \dots & \vdots \\ \xi^1 & \xi^2 & \dots & \xi^n \end{array}\right) = A \left[\begin{array}{cccc} D_1 & D_1 & \dots & D_1 \\ D_2 & D_2 & \dots & D_2 \\ \vdots & \vdots & \dots & \vdots \\ D_n & D_n & \dots & D_n \end{array}\right] \left(\begin{array}{cccc} \xi^1 & \xi^2 & \dots & \xi^n \\ \xi^1 & \xi^2 & \dots & \xi^n \\ \vdots & \vdots & \dots & \vdots \\ \xi^1 & \xi^2 & \dots & \xi^n \end{array}\right) \tag{2.30}$$

Now transposing both sides gives,

$$X(A^T) = \left(\begin{array}{cccc} D_1\xi^1 & D_2\xi^1 & \dots & D_n\xi^1 \\ D_1\xi^2 & D_2\xi^2 & \dots & D_n\xi^2 \\ \vdots & \vdots & \vdots & \vdots \\ D_1\xi^n & D_2\xi^n & \dots & D_n\xi^n \end{array}\right) A^T \tag{2.31}$$

Since $A^T(A^{-1})^T = I$, then $X(A^T)(A^{-1})^T + A^T X((A^{-1})^T) = 0$, thus

$$\begin{aligned} X((A^{-1})^T) &= -(A^T)^{-1}X(A^T)(A^{-1})^T = -(A^{-1})^T X(A^T)(A^T)^{-1} \\ &= -(A^{-1})^T \left(\begin{array}{cccc} D_1\xi^1 & D_2\xi^1 & \dots & D_n\xi^1 \\ D_1\xi^2 & D_2\xi^2 & \dots & D_n\xi^2 \\ \vdots & \vdots & \vdots & \vdots \\ D_1\xi^n & D_2\xi^n & \dots & D_n\xi^n \end{array}\right) . \end{aligned} \tag{2.32}$$

Lastly we get the result

$$\left(\begin{array}{c} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{array}\right) = J(A^{-1})^T \left(\left(\begin{array}{c} XT^1 \\ XT^2 \\ \vdots \\ XT^n \end{array}\right) - \left(\begin{array}{cccc} D_1\xi^1 & D_2\xi^1 & \dots & D_n\xi^1 \\ D_1\xi^2 & D_2\xi^2 & \dots & D_n\xi^2 \\ \vdots & \vdots & \vdots & \vdots \\ D_1\xi^n & D_2\xi^n & \dots & D_n\xi^n \end{array}\right) \left(\begin{array}{c} T^1 \\ T^2 \\ \vdots \\ T^n \end{array}\right) + D_i\xi^i \left(\begin{array}{c} T^1 \\ T^2 \\ \vdots \\ T^n \end{array}\right) \right) \tag{2.33}$$

Corollary 2.1 (The necessary and sufficient condition to get reduced conserved form)
The conserved form $D_iT^i = 0$ of PDE system (2.1) can be reduced under the similarity transformation of a symmetry X to a reduced conserved form $\tilde{D}_i\tilde{T}^i = 0$ if and only if X is associated with the conservation law T, i.e. $[T,X]\mid_{(2.1)} = 0$.

Corollary 2.2 (The generalized double reduction theory)
A non linear system of q th order PDEs with n independent and m dependent variables, which admits a nontrivial conserved form that has at least one associated symmetry in every reduction from the n reductions (the first step of double reduction) can be reduced to a non linear system $(q-1)$th order of ODEs .

Corollary 2.3 (The inherited symmetries)
Any symmetry Y for the conserved form $D_iT^i = 0$ of PDE system (2.1) can be transformed under the similarity transformation of a symmetry X for the PDE to the symmetry $\tilde{Y}$ for the PDE $\tilde{D}_i\tilde{T}^i = 0$.

Remark:
There is a possibility to get an associated symmetry with a reduced conserved form by inhering of the non

associated symmetry with the original conserved form. So there is an important useful of the non associated symmetry also in Double reduction.
Finally we conjecture that the reduction under a combination of an associated and a non associated symmetries will give us two PDE one of them is a reduced conserved form and the second is a non reduced conserved form, we can sperate them via the condition $X(\tilde{D}_i\tilde{T}^i)=0$ such that the solution of a reduced conserved form is also a solution of the non reduced conserved form.

3 Application of the generalized double reduction theory to nonlinear (2 + 1) wave equation

The nonlinear (2 + 1) wave equation for arbitrary function $f(u)$ and $g(u)$

$$u_{tt}-(f(u)u_x)_x-(g(u)u_y)_y=0, \tag{3.1}$$

has the the obvious conservation law

$$T=(-u_t, f(u)u_x, g(u)u_y). \tag{3.2}$$

And admits the following four symmetries:

$$\begin{array}{ll} X_1=\frac{\partial}{\partial t}, & X_2=\frac{\partial}{\partial x} \\ X_3=\frac{\partial}{\partial y} & X_4=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}. \end{array} \tag{3.3}$$

We can get a reduced conserved form for the PDE by the associated symmetry which satisfies the following formula

$$\begin{aligned} X\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}-\begin{pmatrix} D_t\xi^t & D_x\xi^t & D_y\xi^t \\ D_t\xi^x & D_x\xi^x & D_y\xi^x \\ D_t\xi^y & D_x\xi^y & D_y\xi^y \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} \\ +(D_t\xi^t+D_x\xi^x+D_y\xi^y)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}=0. \end{aligned} \tag{3.4}$$

Then the only associated symmetries are X_1, X_2 and X_3, so we can get a reduced conserved form by the combination of them $X=\frac{\partial}{\partial t}+c_1\frac{\partial}{\partial x}+c_2\frac{\partial}{\partial y}$, where the generator X has a canonical form $X=\frac{\partial}{\partial q}$ when

$$\frac{dt}{1}=\frac{dx}{c_1}=\frac{dy}{c_2}=\frac{du}{0}=\frac{dr}{0}=\frac{ds}{0}=\frac{dq}{1}=\frac{dw}{0}, \tag{3.5}$$

or

$$r=y-c_2t, \quad s=x-c_1t, \quad q=t, \quad w(r,s)=u. \tag{3.6}$$

Using the following formula, we can get the reduced conserved form

$$\begin{pmatrix} T^r \\ T^s \\ T^q \end{pmatrix}=J(A^{-1})^T\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}, \tag{3.7}$$

where

$$A^{-1}=\begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix}, \quad J=det(A). \tag{3.8}$$

Then the reduced conserved form is

$$D_r T^r + D_s T^s = 0, \tag{3.9}$$

where

$$\begin{aligned} T^r &= {c_2}^2 w_r + c_2 c_1 w_s - g(w) w_r, \\ T^s &= c_1 c_2 w_r + c_1^2 w_s - f(w) w_s, \\ T^q &= -c_2 w_r - c_1 w_s. \end{aligned} \tag{3.10}$$

The reduced conserved form admits the inherited symmetry:

$$\tilde{X}_4 = r\tfrac{\partial}{\partial r} + s\tfrac{\partial}{\partial s}, \tag{3.11}$$

Similarly we can get a reduced conserved form for the PDE by the associated symmetry which satisfies the following formula

$$\begin{aligned} X\begin{pmatrix} T^r \\ T^s \end{pmatrix} - \begin{pmatrix} D_r\xi^r & D_s\xi^r \\ D_r\xi^s & D_s\xi^s \end{pmatrix}\begin{pmatrix} T^r \\ T^s \end{pmatrix} \\ +(D_r\xi^r + D_s\xi^s)\begin{pmatrix} T^r \\ T^s \end{pmatrix} = 0. \end{aligned} \tag{3.12}$$

One can see that $\tilde{X}_4$ is an associated symmetry, so we can get a reduced conserved form by $Y = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}$, where the generator Y has a canonical form $Y = \frac{\partial}{\partial m}$ when

$$\frac{dr}{r} = \frac{ds}{s} = \frac{dw}{0} = \frac{dn}{0} = \frac{dm}{1} = \frac{dv}{0}, \tag{3.13}$$

or

$$n = \tfrac{s}{r}, \quad m = lnr, \quad v(n) = w. \tag{3.14}$$

So by using the following formula, we can get the reduced conserved form

$$\begin{pmatrix} T^n \\ T^m \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^r \\ T^s \end{pmatrix}, \tag{3.15}$$

where

$$A^{-1}=\begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix}, \quad J=det(A). \tag{3.16}$$

Then the reduced conserved form is:

$$D_n T^n = 0, \tag{3.17}$$

where

$$\begin{aligned} T^n &= v_n(-{c_2}^2 n^2 + 2c_2 c_1 n + n^2 g(v) - {c_1}^2 + f(v)), \\ T^m &= -v_n(-c_2^2 n + c_2 c_1 + n g(v)). \end{aligned} \tag{3.18}$$

The second step of double reduction can be given as

$$v_n(-{c_2}^2 n^2 + 2c_2 c_1 n + n^2 g(v) - {c_1}^2 + f(v)) = C, \tag{3.19}$$

where C is a constant, $n = \frac{x - c_1 t}{y - c_2 t}$ and $v = u$.

4 Conclusion

We have shown that the double reduction theory is still true in general case. This shows that one can obtain *the invariant solution for a non linear system of PDEs by this procedure from the association of the symmetry with its conserved form via the new generalized formula (2.11).*

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